APPLICATION OF A VARIATIONAL METHOD FOR THE SOLUTION OF NONSTATIONARY HEAT CONDUCTION PROBLEMS

N. M. Tsirel'man and D. M. Yanbulatov UDC 536.21

We present an application of a variational principle due to Ainola, compatible with Kantorovich's method, for the solution of the first and second boundary value problems.

The majority of papers on variational methods for the solution of nonstationary heat conduction problems involve the use of Biot's integrodifferential principle [1], although the approximations obtained thereby are somewhat involved owing to the absence in these methods of explicit physical bases, (see [2]). We assume here that this deficiency is eliminated in Ainola's variational principle, formulated for simultaneous application with the method of Ritz in [3]. Our aim here is to show that it is possible to apply Ainola's variational principle, along with Kantorovich's method, to obtain approximate analytic solutions of one-dimensional nonhomogeneous heat conduction equations with boundary conditions of the first and second kinds. Ainola's principle is formulated here in a form suitable for practical application; in particular, the functional is composed starting from the heat conduction equation.

Let it be required to solve the heat conduction equation

$$\frac{1}{x^m}\frac{\partial}{\partial x}\left[x^m\lambda(x)\frac{\partial T}{\partial x}\right] = c\rho(x)\frac{\partial T}{\partial \tau} + q_v(x,\ \tau),\quad m = 0,\ 1,\ 2,\qquad a < x < b,\quad \tau > 0 \tag{1}$$

subject to the initial condition

$$T(x,\ 0) = \varphi(x),\qquad a \leqslant x \leqslant b \tag{2}$$

and a boundary condition of the first kind

$$T(a,\ \tau) = f_1(\tau),\qquad T(b,\ \tau) = f_2(\tau),\qquad \tau > 0 \tag{$3_{\rm I}$}$$

or a boundary condition of the second kind

$$\left.\frac{\partial T}{\partial x}\right|_{x=a} = f_1(\tau),\quad \left.\frac{\partial T}{\partial x}\right|_{x=b} = f_2(\tau),\quad \tau > 0,^* \tag{$3_{\rm II}$}$$

where $\varphi(x)$, $f_1(\tau)$, and $f_2(\tau)$ are continuous functions satisfying, for boundary conditions of the first kind, the compatibility conditions

$$\varphi(a) = f_1(0),\qquad \varphi(b) = f_2(0) \tag{$4_{\rm I}$}$$

or, for boundary conditions of the second kind

$$\varphi'(a) = f_1(0),\qquad \varphi'(b) = f_2(0). \tag{$4_{\rm II}$}$$

Introducing a new unknown function $u(x, \tau)$ such that

$$T(x,\ \tau) = u(x,\ \tau) + \varphi(x) + [f_1(\tau) - \varphi(a)]\frac{b-x}{b-a} + [f_2(\tau) - \varphi(b)]\frac{x-a}{b-a}, \tag{$5_{\rm I}$}$$

*Relations referring to boundary conditions of the first or second kinds, respectively, are numbered with a subscript 1 or 2; relations without such subscripts apply to both types of boundary conditions.

Ordzhonikidze Ufa Aeronautical Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 27, No. 1, pp. 138-144, July, 1974. Original article submitted December 7, 1973.

$$T(x,\ \tau)=u(x,\ \tau)+\varphi(x)+[f_1(\tau)-\varphi'(a)]\frac{2bx-x^2}{2(b-a)}+[f_2(\tau)-\varphi'(b)]\frac{x^2-2ax}{2(b-a)} \tag{5$_{\text{II}}$}$$

respectively, for boundary conditions of the first and second kinds, we reduce the problems defined by Eqs. (1), (3_{I}) and (1), (3_{II}) to the zero initial and boundary conditions

$$\frac{1}{x^m}\frac{\partial}{\partial x}\left[x^m\lambda(x)\frac{\partial u}{\partial x}\right]-c\rho(x)\frac{\partial u}{\partial \tau}-f(x,\ \tau)=0, \tag{1'}$$

$$u(x,\ 0)=0,\qquad a\leqslant x\leqslant b, \tag{2'}$$

$$u(a,\ \tau)=0,\qquad u(b,\ \tau)=0,\qquad \tau>0, \tag{3$_{\text{I}}'$}$$

$$u'(a,\ \tau)=0,\qquad u'(b,\ \tau)=0,\qquad \tau>0. \tag{3$_{\text{II}}'$}$$

In Eq. (1') the quantity $f(x,\tau)$, for the boundary conditions (3_{I}) of the first kind, is given by

$$f(x,\ \tau)=q_v(x,\ \tau)+\frac{(b-x)f_1'(\tau)-(x-a)f_2'(\tau)}{b-a}-$$
$$-\frac{1}{x^m}\frac{\partial}{\partial x}\left\{x^m\lambda(x)\left[\varphi'(x)+\frac{f_2(\tau)-f_1(\tau)+\varphi(a)-\varphi(b)}{b-a}\right]\right\} \tag{6$_{\text{I}}$}$$

and, for the boundary conditions of the second kind, (3_{II}), by

$$f(x,\ \tau)=q_v(x,\ \tau)+\frac{f_1'(\tau)(2bx-x^2)+f_2'(\tau)(x^2-2ax)}{2(b-a)}-$$
$$-\frac{1}{x^m}\frac{\partial}{\partial x}\left\{x^m\lambda(x)\left\{\varphi'(x)+\frac{[f_1(\tau)-\varphi'(a)](b-x)+[f_2(\tau)-\varphi'(b)](x-a)}{2(b-a)}\right\}\right\}. \tag{6$_{\text{II}}$}$$

As was done in [3], we integrate the first two terms of (1') by parts, and, using the symmetry property of the kernel, we confirm the validity of the following statement: if $u(x, \tau)$ is a solution of Eq. (1') for $0 < \tau < t$, satisfying the conditions (2'), ($3_{\text{I}}'$) or the conditions (2'), ($3_{\text{II}}'$), then the functional

$$I=\int_a^b\int_0^t\left\{\frac{1}{x^m}\frac{\partial}{\partial x}\left[x^m\lambda(x)\frac{\partial u}{\partial x}\right]-c\rho(x)\frac{\partial u}{\partial\tau}-2f(x,\ \tau)\right\}x^m u(x,\ t-\tau)\,dx\,d\tau \tag{7}$$

has a stationary solution since its variation $\delta I = 0$.

In accordance with Kantorovich's method [4] we write the first approximation to the solution of Eq. (7) in the form

$$u(x,\ \tau)=g(x)\psi(\tau), \tag{8}$$

where $g(x)$ is a known function of the coordinates, satisfying the conditions $g(a) = g(b) = 0$ for boundary conditions of the first kind and the conditions $g'(a) = g'(b) = 0$ for boundary conditions of the second kind, and $\psi(\tau)$ is an unknown function such that $\psi(0) = 0$. We note here that the functions $g(x)$ for the first and succeeding approximations, besides satisfying the homogeneous boundary conditions of the problem and the completeness conditions, must be continuous and differentiable in the domain being studied. The specific form of the functions $g(x)$ is justified and presented, for example, in [5].

Substituting the expression (8) into the functional (7) and integrating with respect to x, we obtain

$$I=\int_0^t[A\psi(\tau)+B\psi'(\tau)-2C(\tau)]\,\psi(t-\tau)\,d\tau, \tag{7'}$$

where

$$A=\int_a^b[x^m\lambda(x)g'(x)]'\,g(x)\,dx,$$

$$B=\int_a^b c\rho(x)\,x^m g^2(x)\,dx,$$

$$C(\tau)=\int_a^b f(x,\ \tau)\,g(x)\,x^m dx.$$

It is easy to show that for the functional (7') the stationary condition (Euler equation) is

$$A\psi(\tau) - B\psi'(\tau) - C(\tau) = 0. \tag{9}$$

Solving this equation, for example, with the aid of the Laplace integral transform and the condition $\psi(0) = 0$, we determine $\psi(\tau)$ and, consequently, also the solution of the problem (1), ($3_I'$) or the problem (1), ($3_{II}'$). The succeeding approximations are obtained in an analogous way.

By way of illustrating the above, we consider the temperature field in homogeneous bodies of the simplest geometrical shape (plate, cylinder, sphere) without heat sources under constant symmetrical boundary conditions and a constant initial condition.

In this case the boundary value problems (1), (3_I) and (1), (3_{II}) have the form

$$\frac{1}{\eta^m}\frac{\partial}{\partial\eta}\left(\eta^m\frac{\partial T}{\partial\eta}\right) = \frac{\partial T}{\partial\bar{\tau}}, \quad 0<\eta<1, \quad \bar{\tau}>0, \tag{1''}$$

$$T(0, \eta) = T_0, \qquad 0 \leqslant \eta \leqslant 1, \tag{2''}$$

$$T(1, \tau) = T_m, \qquad \bar{\tau} > 0, \tag{$3_I''$}$$

$$\left.\frac{\partial T}{\partial\eta}\right|_{\eta=1} = \frac{ql_0}{\lambda}, \qquad \bar{\tau} > 0 \tag{$3_{II}''$}$$

supplemented by the symmetry condition

$$\left.\frac{\partial T}{\partial\eta}\right|_{\eta=0} = 0. \tag{10}$$

To satisfy the compatibility condition (4_I) we put

$$f_1(\bar{\tau}) = T(\pm 1, \bar{\tau}) = \begin{cases} \dfrac{T_0 - T_m}{\bar{\tau}_0^2}(\bar{\tau} - \bar{\tau}_0)^2 + T_m & 0 \leqslant \bar{\tau} \leqslant \bar{\tau}_0, \\ T_m, & \bar{\tau}_0 \leqslant \bar{\tau}, \end{cases}$$

and to satisfy the condition (4_{II})

$$f_1(\bar{\tau}) = \left.\frac{\partial T}{\partial\eta}\right|_{\eta=1} = \begin{cases} \dfrac{ql_0}{\lambda} - \dfrac{ql_0}{\lambda\bar{\tau}_0^2}(\bar{\tau} - \bar{\tau}_0)^2, & 0 \leqslant \bar{\tau} \leqslant \bar{\tau}_0, \\ \dfrac{ql_0}{\lambda}, & \bar{\tau}_0 \leqslant \bar{\tau}, \end{cases}$$

where $\bar{\tau}_0$ is a sufficiently small positive number. The physical meaning of this is that we presuppose the existence of a transition process of duration $\bar{\tau}_0$ in the course of which the initial temperature T_0 on the surface of the body varies continuously up to the temperature T_m of the surrounding medium (for boundary conditions of the first kind) or, in the exterior layers of the body, a temperature profile is formed whose derivative at the wall is equal to $q(l_0/\lambda)$ (for boundary conditions of the second kind). These assumptions are in accord with the actual occurrence of nonstationary heat conduction processes.

We first introduce, in accord with Eq. (5_I), $u(\eta, \bar{\tau})$ as follows:

$$T(\eta, \bar{\tau}) = u(\eta, \bar{\tau}) + \frac{1+\eta}{2} f_1(\bar{\tau}) \tag{$5_I'$}$$

and, in accord with Eq. (5_{II}),

$$T(\eta, \bar{\tau}) = u(\eta, \bar{\tau}) + T_0 + \frac{\eta^2}{2} f_1(\bar{\tau}). \tag{$5_{II}'$}$$

Then, for the first boundary value problem, the functional (7) takes on the form

$$I = \int_0^1\int_0^{\bar{t}}\left[\frac{1}{\eta^m}\frac{\partial}{\partial\eta}\left(\eta^m\frac{\partial u}{\partial\eta}\right) - \frac{\partial u}{\partial\bar{\tau}} - (1+\eta)f_1'(\bar{\tau}) - \frac{m}{\eta}f_1(\bar{\tau})\right]\eta^m u(\eta, \bar{t} - \bar{\tau})\, d\eta d\bar{\tau} \tag{$7_I'$}$$

and, for the other problem,

$$I = \int_0^1 \int_0^{\bar{t}} \left[\frac{1}{\eta^m} \frac{\partial}{\partial \eta} \left(\eta^m \frac{\partial u}{\partial \eta} \right) - \frac{\partial u}{\partial \bar{\tau}} - \eta^2 f_1'(\bar{\tau}) - 2(m+1) f_1(\bar{\tau}) \right] \eta^m u(\eta, \bar{t} - \bar{\tau}) \, d\eta d\bar{\tau}. \qquad (7'_{II})$$

If, following the expression (8), we put $u(\eta, \bar{\tau}) = (1-\eta^2)\psi(\bar{\tau})$ for the first boundary value problem and $-u(\eta, \bar{\tau}) = \psi(\bar{\tau})$ for the second, and carry out the integrations in Eqs. $(7'_I)$ and $(7'_{II})$ with respect to η, we obtain functionals with the respective Euler equations

$$\psi'(\bar{\tau}) + \frac{(m+1)(m+5)}{2} \psi(\bar{\tau}) = \frac{m+5}{4} f_1'(\bar{\tau}), \qquad (11_I)$$

$$\psi'(\bar{\tau}) = (m+1) f_1(\bar{\tau}) - \frac{m+1}{2(m+3)} f_1'(\bar{\tau}). \qquad (11_{II})$$

Solving Eqs. (11_I) and (11_{II}) for $\psi(\bar{\tau})$ and using the condition $\psi(0) = 0$, we obtain, as $\bar{\tau}_0 \to 0$, the solution of the first boundary value problem

$$\theta(\eta, \bar{\tau}) = \frac{T(\eta, \bar{\tau}) - T_m}{T_0 - T_m} = \frac{m+5}{4}(1-\eta^2) \exp\left[-\frac{(m+1)(m+5)}{2} \bar{\tau} \right] \qquad (12_I)$$

and the solution of the second boundary value problem

$$\theta(\eta, \bar{\tau}) = \frac{T(\eta, \bar{\tau}) - T_0}{\frac{ql_0}{\lambda}} = (m+1)\bar{\tau} - \frac{m+1-(m+3)\eta^2}{2(m+3)}, \qquad (12_{II})$$

where the solution (12_{II}) agrees in full with the quasistationary part of the exact solution given in [6].

We seek second approximations $u(\eta, \bar{\tau})$ in the respective forms

$$u(\eta, \bar{\tau}) = (1-\eta^2)\psi_1(\bar{\tau}) + (1-\eta^2)\eta^2\psi_2(\bar{\tau}), \qquad (13_I)$$

$$u(\eta, \bar{\tau}) = \psi_1(\bar{\tau}) + \left(\frac{\eta^2}{2} - \frac{\eta^4}{4} \right) \psi_2(\bar{\tau}) \qquad (13_{II})$$

and, omitting for lack of space, the systems of Euler equations for the determination of $\psi_1(\bar{\tau})$ and $\psi_2(\bar{\tau})$, we give, for the sake of comparison with the solutions in [6], only the magnitudes of θ as $\bar{\tau}_0 \to 0$.

For the first boundary value problem we write the second approximation to the solution as follows:

$$\begin{aligned} \theta = \frac{(m+7)(1-\eta^2)}{16\sqrt{(m+3)(m+7)(m^2+10m+57)}} \{ [(\beta - \alpha_1)(3-m) - \\ -(m+9)\alpha_1\eta^2] \exp(-\alpha_1\bar{\tau}) + [\alpha_2(m+9)\eta^2 - (3-m)(\beta-\alpha_2)] \exp(-\alpha_2\bar{\tau}) \}, \end{aligned} \qquad (14_I)$$

where

$$\left.\begin{matrix}\alpha_1 \\ \alpha_2\end{matrix}\right\} = \frac{(m+3)(m+7)}{3} \left[2 \mp \sqrt{\frac{m^2+10m+57}{(m+3)(m+7)}} \right], \quad \beta = \frac{4(m+3)(m+9)}{3-m}.$$

For the second boundary value problem we have, in the second approximation, as $\bar{\tau}_0 \to 0$,

$$\begin{aligned} \theta = (m+1)\bar{\tau} + \frac{(m+3)\eta^2 - m - 1}{2(m+3)} + [(m+1)(m+7) - \\ - 2(m+3)(m+5)\eta^2 + (m+3)(m+5)\eta^4] \frac{m+9}{4(m+3)(5m+27)} \exp(-\gamma\bar{\tau}), \end{aligned} \qquad (14)$$

where

$$\gamma = \frac{2(m+3)(m+5)(m+9)}{5m+27}.$$

We note that the quantities α_1 and α_2 are practically coincident with the squares of the first and second characteristic numbers for the first boundary value problem of heat conduction, while the quantity γ coincides with the square of the first characteristic number for the second boundary value problem.

The solution of the first boundary value problem for the particular case in which $\lambda = \lambda_0(1 + \nu\eta^2)$, $c\rho = c_0\rho_0(1 + \chi\eta)$, and $q_V(\eta, \tau) = q_0 \exp(-k\tau)$, is, in the first approximation as $\bar{\tau}_0 \to 0$, as follows:

$$\theta(\eta, \bar{\tau}) = \frac{1-\eta^2}{2M}\left[N \exp(-\mu\bar{\tau}) + \mathrm{Po}\frac{\exp(-\mu\bar{\tau}) - \exp(-\mathrm{Pd}\cdot\bar{\tau})}{(m+1)(m+3)(\mathrm{Pd}-\mu)}\right],$$

where

$$M = \frac{2[(m+2)(m+4)(m+6) + (m+1)(m+3)(m+5)\chi]}{(m+1)(m+2)(m+3)(m+4)(m+5)(m+6)};$$

$$N = \frac{(m+2)(m+4) + (m+1)(m+3)\chi}{(m+1)(m+2)(m+3)(m+4)}, \quad \mu = \frac{m+5+(m+3)\nu}{(m+3)(m+5)M};$$

$$\mathrm{Po} = \frac{q_0 l_0^2}{\lambda_0(T_0 - T_c)}, \quad \mathrm{Pd} = \frac{k l_0^2}{a q_0}.$$

The error of the method considered here is determined by comparing the quantities θ for the second approximation with the exact values θ_e, given in [6] for various $\bar{\tau}$ = idem and η = idem. For the homogeneous first boundary value problem an analysis of the function θ as to monotonicity in the region $|\eta| < 0.5$, along with calculations on an electronic digital computer, show that there is good agreement between θ and θ_e in the region

$$\bar{\tau} > \frac{1}{\alpha_2 - \alpha_1}\left[\frac{3}{2}\ln\frac{\alpha_2}{\alpha_1} + \ln\frac{(3-m)(\alpha_2 - \beta) + \alpha_2(m+9)\eta^2}{(3-m)(\alpha_1 - \beta) + \alpha_1(m+9)\eta^2}\right].$$

For $|\eta| \geq 0.5$ the difference between θ and θ_e is less than 5% for $\bar{\tau} \geq 0.05$.

Calculations carried out on the M-220 electronic digital computer confirm that the quantities θ for the second boundary value problem differ from the corresponding θ_e by less than 5% for $\bar{\tau} \geq 0.03$ for $|\eta| \geq 0.5$, and for $\bar{\tau} \geq 0.1$ for $|\eta| \leq 0.5$.

NOTATION

$T(x, \tau)$, T_0, T_m	are the instantaneous, initial temperature of body and ambient temperature;
x, $\eta = x/l_0$, l_0	are the dimensional and dimensionless coordinates, characteristic dimensions of body (half thickness of plate, radius of cylinder, and sphere);
τ, t, $\bar{\tau} = a\tau/l_0^2$, $\bar{t} = at/l_0^2$	are the dimensional and dimensionless time;
$a = \lambda/c\rho$, λ, c, ρ	are the thermal diffusivity, thermal conductivity, heat capacity, and density;
$q_V(x, \tau)$	is the volume power of heat source;
$m = 0, 1, 2$	are for plate, cylinder, and sphere, respectively;
q	is the heat flux density;
$\mathrm{Po} = q_0 l_0^2/\lambda_0(T_0 - T_m)$	is the Pomerantsev number;
$\mathrm{Pd} = k l_0^2/a q_0$	is the Predvoditelev number.

LITERATURE CITED

1. M. A. Biot, Variational Principles in Heat Transfer, Clarendon Press, Oxford (1970).
2. R. S. Schechter, Variational Method in Engineering, McGraw-Hill, New York (1967).
3. L. Ya. Ainola, Inzh.-Fiz. Zh., 12, No. 4 (1967).
4. L. V. Kantorovich and V. I. Krylov, Approximate Methods of Higher Analysis, Wiley-Interscience, New York (1959).
5. P. V. Tsoi, Calculational Methods for Individual Heat- and Mass Transfer Problems [in Russian], Énergiya, Moscow (1971).
6. A. V. Lykov, Analytical Heat Diffusion Theory, Academic, New York (1968).